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Half Inverse Problem for the Impulsive Diffusion Operator with Discontinuous Coefficient

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Abstract. The half inverse problem is to construct coefficients of the operator in a whole interval by using one spectrum and potential known in a semi interval. In this paper, by using the Hocstadt-Lieberman and Yang-Zettl's methods we show that if $p(x)$ and $q(x)$ are known on the interval $(\pi/2, \pi)$, then only one spectrum suffices to determine $p(x)$, $q(x)$ functions and β, h coefficients on the interval $(0, \pi)$ for impulsive diffusion operator with discontinuous coefficient.

1. Introduction

Inverse spectral problem is recovering the operator from its given spectral datas. These problems are of great importance in applied mathematics and physics, for example, vibration of a string, quantum mechanics etc. Inverse spectral problems for regular or singular Sturm-Liouville and diffusion operators are investigated in $[1-32]$.

First results on half inverse problems for regular Sturm-Liouville operator were given by Hochstadt and Lieberman in [33]. In later years, half inverse problems for various Sturm-Liouville operators and diffusion operators, i.e., quadratic pencils of Sturm-Liouville operators, were studied by authors $[34-44]$.

In this paper, we denote the problem $L = L(p, q, \alpha, \beta, \gamma, h, H)$ of the form

$$\ell y(x) = -y''(x) + [2\lambda p(x) + q(x)]\, y(x) = \lambda^2 \rho(x)\, y(x), \; x \in (0, \pi) \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \; V(y) := y'(\pi) + Hy(\pi) = 0 \tag{2}$$

and the discontinuity conditions

$$\begin{cases} y\left(\dfrac{\pi}{2}+0\right) = \beta y\left(\dfrac{\pi}{2}-0\right) \\ y'\left(\dfrac{\pi}{2}+0\right) = \beta^{-1} y'\left(\dfrac{\pi}{2}-0\right) + \gamma y\left(\dfrac{\pi}{2}-0\right) \end{cases} \tag{3}$$

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where real-valued functions $p(x) \in W_2^1(0,\pi)$, $q(x) \in L_2(0,\pi)$, λ is the spectral parameter, α, β, γ are real numbers, $\beta > 0$, $\left|\beta - 1\right|^2 + \gamma^2 \neq 0$ and

$$\rho(x) = \begin{cases} 1, & 0 < x < \dfrac{\pi}{2} \\ \alpha^2, & \dfrac{\pi}{2} < x < \pi \end{cases}, 0 < \alpha \neq 1.$$

To study the half inverse problem, we consider a boundary value problem $\widetilde{L}$, together with L, of the same form but with different coefficients $\widetilde{p}(x)$, $\widetilde{q}(x)$, $\widetilde{h}$, α, γ and $\widetilde{\beta}$. Hence, we consider a second problem $\widetilde{L} = L\left(\widetilde{p}, \widetilde{q}, \alpha, \widetilde{\beta}, \gamma, \widetilde{h}, H\right)$ of the form

$$\widetilde{\ell} y(x) = -y''(x) + \left[2\lambda \widetilde{p}(x) + \widetilde{q}(x)\right] y(x) = \lambda^2 \rho(x) y(x), \ x \in (0,\pi) \tag{4}$$

with the boundary conditions

$$U(y) := y'(0) - \widetilde{h} y(0) = 0, \ V(y) := y'(\pi) + H y(\pi) = 0 \tag{5}$$

and the discontinuity conditions

$$\begin{cases} y\left(\dfrac{\pi}{2} + 0\right) = \widetilde{\beta} y\left(\dfrac{\pi}{2} - 0\right) \\ y'\left(\dfrac{\pi}{2} + 0\right) = \widetilde{\beta}^{-1} y'\left(\dfrac{\pi}{2} - 0\right) + \gamma y\left(\dfrac{\pi}{2} - 0\right). \end{cases} \tag{6}$$

The aim of this paper is to solve half inverse problem for the problem L by using the Hocstadt-Lieberman and Yang-Zettl's methods. That is, we proved that if $p(x)$ and $q(x)$ functions are known on the interval $(\pi/2, \pi)$, then only one spectrum suffices to determine $p(x)$, $q(x)$ functions and β, h coefficients on the interval $(0,\pi)$ for impulsive diffusion operator with discontinuous coefficient of problem L.

2. Preliminaries

Let $\varphi(x,\lambda)$ be the solution of equation (1) satisfying the initial conditions $\varphi(0,\lambda) = 1, \varphi'(0,\lambda) = 0$. There are the functions $A(x,t)$ and $B(x,t)$ whose first order partial derivatives are summable on $(0,\pi)$ for each $x \in (0,\pi)$. The following represantation for $\varphi(x,\lambda)$ solution can be obtained from the appendix

$$\begin{aligned} \varphi(x,\lambda) = \ & \beta^+ \cos\left(\lambda \mu^+(x) - \frac{\omega(x)}{\sqrt{\rho(x)}}\right) + \beta^- \cos\left(\lambda \mu^-(x) + \frac{\omega(x)}{\sqrt{\rho(x)}}\right) \\ & + \int_0^{\mu^+(x)} A(x,t) \cos \lambda t dt + \int_0^{\mu^+(x)} B(x,t) \sin \lambda t dt \end{aligned} \tag{7}$$

where $\beta^{\pm} = \dfrac{1}{2}\left(\beta \pm \dfrac{1}{\alpha\beta}\right)$, $\mu^{\pm}(x) = \pm\sqrt{\rho(x)}x + \dfrac{\pi}{2}\left(1 \mp \sqrt{\rho(x)}\right)$, $\omega(x) = \displaystyle\int_0^x p(t) dt$.

It is easy to verify from the integral representation above that the following asymptotic relation is valid for $|\lambda| \to \infty$

$$\varphi(x,\lambda) \ = \beta^+ \cos\left(\lambda \mu^+(x) - \frac{\omega(x)}{\sqrt{\rho(x)}}\right) + \beta^- \cos\left(\lambda \mu^-(x) + \frac{\omega(x)}{\sqrt{\rho(x)}}\right) + O\left(\frac{\exp \tau \mu^+(x)}{\lambda}\right) \tag{8}$$

where $\tau := |\mathrm{Im}\,\lambda|$.

The function

$$\Delta(\lambda) := V(\varphi) = \varphi'(\pi,\lambda) + H\varphi(\pi,\lambda) \tag{9}$$

is called the characteristic function for the problem L. Since the boundary value problem L is self-adjoint, all zeros of $\Delta(\lambda)$ are real and simple under the following conditions

$$y'(0)\,\overline{y(0)} - y'(\pi)\,\overline{y(\pi)} = 0$$

and

$$\int_0^\pi \left\{ \left|y'(x)\right|^2 + q(x)\left|y(x)\right|^2 \right\} dx > 0.$$

From (8) and (9), we have

$$\Delta(\lambda) = \Delta_0(\lambda) + O\left(\frac{\exp \tau\mu^+(\pi)}{\lambda}\right) \tag{10}$$

where

$$\begin{aligned}\Delta_0(\lambda) &= -\beta^+\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + \beta^-\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\\ &\quad + H\beta^+\cos\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + H\beta^-\cos\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right).\end{aligned}$$

The function $\Delta(\lambda)$ is entire in λ. Zeros $\{\lambda_n\}_{n\geq 0}$ of $\Delta(\lambda)$ coincide with the eigenvalues of the problem L. We note that for $\lambda \in \{\lambda : |\lambda - \lambda_n| > \delta\}$ for fixed $\delta > 0$,

$$\Delta(\lambda) \geq (\beta^+|\lambda\alpha| - C)\exp(\tau\mu^+(\pi)). \tag{11}$$

3. Main Result

In this section, we consider the following half inverse problem by using Hochstadt-Lieberman and Yang-Zettl's methods in [33, 40] for problem L.

Lemma 3.1. *If $\lambda_n = \widetilde{\lambda_n}$ for all $n \in \mathbb{N}$ then $\beta = \widetilde{\beta}$.*

Proof. Since $\lambda_n = \widetilde{\lambda_n}$ and $\Delta(\lambda), \widetilde{\Delta}(\lambda)$ are entire functions in λ of order 1 by Hadamard factorization theorem,

$$\Delta(\lambda) = Ce^{a\lambda}\widetilde{\Delta}(\lambda)$$

for all $\lambda \in \mathbb{C}$.

Letting $|\lambda| \to \infty$ for all imaginary λ's, we conclude from

$$\lim_{|\lambda|\to\infty} \frac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)} = \frac{\beta^+}{\widetilde{\beta}^+} e^{i\left(\frac{\omega(\pi)-\widetilde{\omega}(\pi)}{\alpha}\right)}$$

that

$$a = 0, C = \frac{\beta^+}{\widetilde{\beta}^+} e^{i\left(\frac{\omega(\pi)-\widetilde{\omega}(\pi)}{\alpha}\right)},$$

thus

$$\Delta(\lambda) = C\widetilde{\Delta}(\lambda). \tag{12}$$

On the other hand, (12) can be written as

$$\Delta_0(\lambda) - C\widetilde{\Delta}_0(\lambda) = C\left(\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)\right) - (\Delta(\lambda) - \Delta_0(\lambda)). \tag{13}$$

Hence,

$$\begin{aligned}
&C\left(\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)\right) - (\Delta(\lambda) - \Delta_0(\lambda)) \\
&= -\beta^+\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + \beta^-\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right) \\
&+H\beta^+\cos\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + H\beta^-\cos\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right) \\
&-C\left[-\widetilde{\beta}^+\left(\lambda\alpha - \tfrac{\widetilde{p}(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \tfrac{\widetilde{\omega}(\pi)}{\alpha}\right) + \widetilde{\beta}^-\left(\lambda\alpha - \tfrac{\widetilde{p}(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \tfrac{\widetilde{\omega}(\pi)}{\alpha}\right)\right. \\
&\left.+H\widetilde{\beta}^+\cos\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + H\widetilde{\beta}^-\cos\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\right].
\end{aligned} \tag{14}$$

If we multiply both sides of (14) with $\sin\left(\lambda\mu^+(\pi) - \frac{\omega(\pi)}{\alpha}\right)$ and integrate with respect to λ in $(0, T)$ for any positive real number T, then we get

$$\begin{aligned}
&\int_0^T \left[C\left(\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)\right) - (\Delta(\lambda) - \Delta_0(\lambda))\right]\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right)d\lambda \\
&= \int_0^T \left[-\beta^+\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + \beta^-\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\right. \\
&\left.+H\beta^+\cos\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + H\beta^-\cos\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\right]\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right)d\lambda \\
&-C\int_0^T \left[-\widetilde{\beta}^+\left(\lambda\alpha - \tfrac{\widetilde{p}(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \tfrac{\widetilde{\omega}(\pi)}{\alpha}\right) + \widetilde{\beta}^-\left(\lambda\alpha - \tfrac{\widetilde{p}(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \tfrac{\widetilde{\omega}(\pi)}{\alpha}\right)\right. \\
&\left.+H\widetilde{\beta}^+\cos\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + H\widetilde{\beta}^-\cos\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\right]\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right)d\lambda.
\end{aligned} \tag{15}$$

Since $\left(\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)\right) = O(1)$ and $(\Delta(\lambda) - \Delta_0(\lambda)) = O(1)$ for λ in $(0, T)$,

$$\frac{C\alpha\widetilde{\beta}^+}{4}\cos\left(\frac{\omega(\pi) - \widetilde{\omega}(\pi)}{\alpha}\right) - \frac{\alpha\beta^+}{4} = O\left(\frac{1}{T}\right). \tag{16}$$

By letting $T \to \infty$ we conclude with

$$C\cos\left(\frac{\omega(\pi) - \widetilde{\omega}(\pi)}{\alpha}\right) = \frac{\beta^+}{\widetilde{\beta}^+}. \tag{17}$$

Similarly, if we multiply both sides of (14) with $\sin\left(\lambda\mu^-(\pi) + \frac{\omega(\pi)}{\alpha}\right)$ and integrate again with respect to λ in $(0, T)$, then we get

$$C\cos\left(\frac{\omega(\pi) - \widetilde{\omega}(\pi)}{\alpha}\right) = \frac{\beta^-}{\widetilde{\beta}^-}. \tag{18}$$

Taking $\beta > 0$ into account, (17) and (18) implies that $\beta = \widetilde{\beta}$. □

Theorem 3.2. *Let $\{\lambda_n\}$ be the spectrum of both L and $\widetilde{L}$. If $p(x)=\widetilde{p}(x)$ and $q(x)=\widetilde{q}(x)$ on $\left(\frac{\pi}{2},\pi\right)$, then $h=\widetilde{h}$, $\beta=\widetilde{\beta}$, $p(x)=\widetilde{p}(x)$ and $q(x)=\widetilde{q}(x)$ almost everywhere on $(0,\pi)$.*

Proof. It is clear from [24] that the solutions $\varphi(x,\lambda)$, $\widetilde{\varphi}(x,\lambda)$ of equations (1) and (4), respectively, with the initial conditions $\varphi(0,\lambda)=\widetilde{\varphi}(0,\lambda)=1, \varphi'(0,\lambda)=h, \widetilde{\varphi}'(0,\lambda)=\widetilde{h}$ can be expressed in the integral forms on $\left(0,\frac{\pi}{2}\right)$,

$$\varphi(x,\lambda)=\cos(\lambda x-\omega(x))+\int_0^x A(x,t)\cos\lambda t dt+\int_0^x B(x,t)\sin\lambda t dt \tag{19}$$

$$\widetilde{\varphi}(x,\lambda)=\cos(\lambda x-\widetilde{\omega}(x))+\int_0^x \widetilde{A}(x,t)\cos\lambda t dt+\int_0^x \widetilde{B}(x,t)\sin\lambda t dt. \tag{20}$$

where the kernels $\widetilde{A}(x,t)$, $\widetilde{B}(x,t)$ have properties similar to those of $A(x,t)$, $B(x,t)$.

Using (19) and (20), we find that

$$\begin{aligned}
\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda) &= \frac{1}{2}\left[\cos(2\lambda x-\theta(x))+\cos(\omega(x)-\widetilde{\omega}(x))\right]\\
&+\int_0^x A(x,t)\cos(\lambda x-\widetilde{\omega}(x))\cos\lambda t dt+\int_0^x \widetilde{A}(x,t)\cos(\lambda x-\omega(x))\cos\lambda t dt\\
&+\int_0^x B(x,t)\sin\lambda t\cos(\lambda x-\widetilde{\omega}(x))dt+\int_0^x \widetilde{B}(x,t)\sin\lambda t\cos(\lambda x-\omega(x))dt\\
&+\left(\int_0^x A(x,t)\cos\lambda t dt\right)\left(\int_0^x \widetilde{A}(x,t)\cos\lambda t dt\right)+\left(\int_0^x B(x,t)\sin\lambda t dt\right)\left(\int_0^x \widetilde{B}(x,t)\sin\lambda t dt\right)\\
&+\left(\int_0^x A(x,t)\cos\lambda t dt\right)\left(\int_0^x \widetilde{B}(x,t)\sin\lambda t dt\right)+\left(\int_0^x \widetilde{A}(x,t)\cos\lambda t dt\right)\left(\int_0^x B(x,t)\sin\lambda t dt\right)
\end{aligned}$$

where $\theta(x)=\omega(x)+\widetilde{\omega}(x)$.

By extending the range of $A(x,t)$, $\widetilde{A}(x,t)$ evenly and $B(x,t)$, $\widetilde{B}(x,t)$ oddly with respect to the argument t, we can write

$$\begin{aligned}
\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda) &= \frac{1}{2}\left[\cos(2\lambda x-\theta(x))+\cos(\omega(x)-\widetilde{\omega}(x))\right]\\
&+\frac{1}{2}\left[\int_0^x H_c(x,t)\cos(2\lambda t-\theta(t))dt-\int_0^x H_s(x,t)\sin(2\lambda t-\theta(t))dt\right]
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
H_c(x,t) &= 2A(x,x-2t)\cos[\theta(t)-\widetilde{\omega}(x)]+2\widetilde{A}(x,x-2t)\cos[\theta(t)-\omega(x)]\\
&-2B(x,x-2t)\sin[\theta(t)-\widetilde{\omega}(x)]-2\widetilde{B}(x,x-2t)\sin[\theta(t)-\omega(x)]\\
&+K_1(x,t)\cos\theta(t)-K_2(x,t)\cos\theta(t)-M_1(x,t)\sin\theta(t)+M_2(x,t)\sin\theta(t),
\end{aligned}$$

$$H_s(x,t) = 2A(x,x-2t)\sin[\theta(t)-\widetilde{\omega}(x)] + 2\widetilde{A}(x,x-2t)\sin[\theta(t)-\omega(x)]$$
$$+2B(x,x-2t)\cos[\theta(t)-\widetilde{\omega}(x)] + 2\widetilde{B}(x,x-2t)\cos[\theta(t)-\omega(x)]$$
$$+K_1(x,t)\sin\theta(t) - K_2(x,t)\sin\theta(t) + M_1(x,t)\cos\theta(t) - M_2(x,t)\cos\theta(t),$$

$$K_1(x,t) = \int_{-x}^{x-2t} A(x,s)\widetilde{A}(x,s+2t)\,ds + \int_{2t-x}^{x} A(x,s)\widetilde{A}(x,s-2t)\,ds,$$

$$K_2(x,t) = -\int_{-x}^{x-2t} B(x,s)\widetilde{B}(x,s+2t)\,ds - \int_{2t-x}^{x} B(x,s)\widetilde{B}(x,s-2t)\,ds,$$

$$M_1(x,t) = \int_{-x}^{x-2t} A(x,s)\widetilde{B}(x,s+2t)\,ds - \int_{2t-x}^{x} A(x,s)\widetilde{B}(x,s-2t)\,ds,$$

$$M_2(x,t) = -\int_{-x}^{x-2t} B(x,s)\widetilde{A}(x,s+2t)\,ds + \int_{2t-x}^{x} B(x,s)\widetilde{A}(x,s-2t)\,ds.$$

Now, let us write the equations

$$-\varphi''(x,\lambda) + [2\lambda p(x) + q(x)]\varphi(x,\lambda) = \lambda^2\rho(x)\varphi(x,\lambda) \tag{22}$$

and

$$-\widetilde{\varphi}''(x,\lambda) + [2\lambda\widetilde{p}(x) + \widetilde{q}(x)]\widetilde{\varphi}(x,\lambda) = \lambda^2\rho(x)\widetilde{\varphi}(x,\lambda). \tag{23}$$

First, by multiplying (22) with $\widetilde{\varphi}(x,\lambda)$ and (23) with $\varphi(x,\lambda)$, second subtracting them side by side and then integrating on $(0,\pi)$, we get

$$\int_0^{\pi/2} [2\lambda(\widetilde{p}(x)-p(x)) + (\widetilde{q}(x)-q(x))]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx = \left(\widetilde{\varphi}'(x,\lambda)\varphi(x,\lambda) - \varphi'(x,\lambda)\widetilde{\varphi}(x,\lambda)\right)\Big|_0^{\pi/2} + \Big|_{\pi/2}^{\pi}$$

from the hypothesis $p(x) = \widetilde{p}(x)$, $q(x) = \widetilde{q}(x)$ on $\left(\frac{\pi}{2},\pi\right)$ and the initial conditions $\varphi(0,\lambda) = 1$, $\varphi'(0,\lambda) = 0$, we obtain

$$\int_0^{\pi/2} [2\lambda(\widetilde{p}(x)-p(x)) + (\widetilde{q}(x)-q(x))]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx + \widetilde{h} - h + \varphi'(\pi,\lambda)\widetilde{\varphi}(\pi,\lambda) - \widetilde{\varphi}'(\pi,\lambda)\varphi(\pi,\lambda) = 0. \tag{24}$$

Let

$$P(x) := \widetilde{p}(x) - p(x),\ Q(x) := \widetilde{q}(x) - q(x)$$

and

$$H(\lambda) := \widetilde{h} - h + \int_0^{\pi/2} (2\lambda P(x) + Q(x))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx.$$

It is clear from the properties of $\varphi(x,\lambda)$, $\varphi'(x,\lambda)$ and the boundary conditions (2) that the first term in (24) vanishes and thus

$$H(\lambda_n) = 0 \tag{25}$$

for each eigenvalue λ_n.
Let us define

$$H_1(\lambda) = \int_0^{\pi/2} P(x)\,\varphi(x,\lambda)\,\widetilde{\varphi}(x,\lambda)\,dx,\ H_2(\lambda) = \int_0^{\pi/2} Q(x)\,\varphi(x,\lambda)\,\widetilde{\varphi}(x,\lambda)\,dx,$$

then equation (25) can be rewritten as

$$\left(\widetilde{h}-h\right) + 2\lambda_n H_1(\lambda_n) + H_2(\lambda_n) = 0. \tag{26}$$

From (21) and (24), we obtain

$$|H(\lambda)| \le (C_1 + C_2|\lambda|)\exp(\tau\pi) \tag{27}$$

for all complex λ, where $C_1, C_2 > 0$ is constant.
If we denote

$$\Phi(\lambda) := \frac{H(\lambda)}{\Delta(\lambda)}, \tag{28}$$

then $\Phi(\lambda)$ is an entire function with respect to λ.
It follows from (11) and (27) that

$$\Phi(\lambda) = O(1) \tag{29}$$

for sufficiently large $|\lambda|$.
Using Liouville's Theorem, we obtain

$$\Phi(\lambda) = C,\ \text{ for all } \lambda$$

where C is a constant.
Now, we can rewrite the equation $H(\lambda) = C\Delta(\lambda)$ as

$$\left(\widetilde{h}-h\right) + \int_0^{\pi/2} (2\lambda P(x) + Q(x))\,\varphi(x,\lambda)\,\widetilde{\varphi}(x,\lambda)\,dx =$$
$$C\left\{-\beta^+\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + \beta^-\left(\lambda\alpha - \tfrac{p(\pi)}{\alpha}\right)\sin\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\right.$$
$$\left.+H\beta^+\cos\left(\lambda\mu^+(\pi) - \tfrac{\omega(\pi)}{\alpha}\right) + H\beta^-\cos\left(\lambda\mu^-(\pi) + \tfrac{\omega(\pi)}{\alpha}\right)\right\} + O\left(\exp\left(\tau\mu^+(\pi)\right)\right).$$

By the Riemann-Lebesgue Lemma, the limit of the left side of the above equality exists for $\lambda \to \infty$, $\lambda \in \mathbb{R}$. Therefore, we get that $C = 0$. Then

$$\left(\widetilde{h}-h\right) + 2\lambda H_1(\lambda) + H_2(\lambda) = 0\ \text{ for all } \lambda. \tag{30}$$

By virtue of (21),

$$2H_1(\lambda) = \int_0^{\pi/2} P(x)\cos\left(\omega(x) - \widetilde{\omega}(x)\right)dx + \int_0^{\pi/2} P_1(t)\cos\left(2\lambda t - \theta(t)\right)dt - \int_0^{\pi/2} P_2(t)\sin\left(2\lambda t - \theta(t)\right)dt,$$

where

$$P_1(t) = P(t) + \int_t^{\pi/2} P(x) H_c(x,t)\,dx, \quad P_2(t) = \int_t^{\pi/2} P(x) H_s(x,t)\,dx. \tag{31}$$

If we change the order of integration, apply partial integration and take $P_1(\pi/2) = P(\pi/2)$ and $P_2(\pi/2) = 0$ into account, we get

$$\begin{aligned} 2H_1(\lambda) = & \int_0^{\pi/2} P(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx + \int_0^{\pi/2} T_1(t) e^{2i\lambda t}dt - \int_0^{\pi/2} T_2(t) e^{-2i\lambda t}dt \\ = & \int_0^{\pi/2} P(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx + \frac{P(\pi/2)}{2\lambda}\sin[\lambda\pi - \theta(\pi/2)] \\ & - \frac{P_2(0)}{2\lambda} + \frac{i}{2\lambda}\int_0^{\pi/2} T_1'(t) e^{2i\lambda t}dt - \frac{i}{2\lambda}\int_0^{\pi/2} T_2'(t) e^{-2i\lambda t}dt, \end{aligned} \tag{32}$$

where

$$T_1(t) = \frac{P_1(t) + iP_2(t)}{2} e^{-i\theta(t)}, \quad T_2(t) = \frac{P_1(t) - iP_2(t)}{2} e^{i\theta(t)}.$$

Similarly, we get

$$2H_2(\lambda) = \int_0^{\pi/2} Q(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx + \int_0^{\pi/2} Q_1(t)\cos(2\lambda t - \theta(t))\,dt - \int_0^{\pi/2} Q_2(t)\sin(2\lambda t - \theta(t))\,dt,$$

where

$$Q_1(t) = Q(t) + \int_t^{\pi/2} Q(x) H_c(x,t)\,dx, \quad Q_2(t) = \int_t^{\pi/2} Q(x) H_s(x,t)\,dx. \tag{33}$$

By changing the order of integration, we obtain

$$2H_2(\lambda) = \int_0^{\pi/2} Q(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx + \int_0^{\pi/2} R_1(t) e^{2i\lambda t}dt + \int_0^{\pi/2} R_2(t) e^{-2i\lambda t}dt \tag{34}$$

where

$$R_1(t) = \frac{Q_1(t) + iQ_2(t)}{2} e^{-i\theta(t)}, \quad R_2(t) = \frac{Q_1(t) - iQ_2(t)}{2} e^{i\theta(t)}.$$

If (32) and (34) are substituted into (30), we get

$$\begin{aligned} & \left(\widetilde{h} - h\right) + 2\lambda\int_0^{\pi/2} P(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx + \int_0^{\pi/2} Q(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx + P(\pi/2)\sin(\lambda\pi - \theta(\pi/2)) \\ & - P_2(0) + i\int_0^{\pi/2} T_1'(t) e^{2i\lambda t}dt - i\int_0^{\pi/2} T_2'(t) e^{-2i\lambda t}dt + \int_0^{\pi/2} R_1(t) e^{2i\lambda t}dt + \int_0^{\pi/2} R_2(t) e^{-2i\lambda t}dt = 0. \end{aligned} \tag{35}$$

Using the Riemann-Lebesgue Lemma for $\lambda \to \infty$, $\lambda \in \mathbb{R}$ in (35), then it follows that

$$\begin{cases} \int_0^{\pi/2} P(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx = 0 \\ P(\pi/2) = 0 \\ 2\left(\widetilde{h} - h\right) + \int_0^{\pi/2} Q(x)\cos(\omega(x) - \widetilde{\omega}(x))\,dx = 0 \end{cases} \tag{36}$$

and

$$\int_0^{\pi/2} \left(R_1(t) + iT_1'(t)\right)e^{2i\lambda t}dt + \int_0^{\pi/2} \left(R_2(t) - iT_2'(t)\right)e^{-2i\lambda t}dt = 0$$

for all complex number λ.

Since the system $\left\{e^{\pm 2i\lambda t} : \lambda \in \mathbb{R}\right\}$ is entire in $L_2\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows

$$\begin{cases} R_1(t) + iT_1'(t) = 0 \\ R_2(t) - iT_2'(t) = 0 \end{cases}$$

which yields the following system

$$\begin{cases} \left(Q_1(t) + P_1(t)\theta'(t) - P_2'(t)\right) + i\left(Q_2(t) + P_2(t)\theta'(t) + P_1'(t)\right) = 0 \\ \left(Q_1(t) + P_1(t)\theta'(t) - P_2'(t)\right) - i\left(Q_2(t) + P_2(t)\theta'(t) + P_1'(t)\right) = 0 \end{cases}.$$

And hence,

$$\begin{cases} Q_1(t) + P_1(t)\theta'(t) - P_2'(t) = 0 \\ Q_2(t) + P_2(t)\theta'(t) + P_1'(t) = 0 \end{cases}. \tag{37}$$

Substituting (31) and (33) into system (37) and taking $P(\pi/2) = 0$ into account, it yields

$$\begin{cases} Q(t) &= -\int_t^{\pi/2} H_c(x,t)Q(x)\,dx - \int_t^{\pi/2} \left(\theta'(t)H_c(x,t) - \dfrac{\partial H_s(x,t)}{\partial t}\right)P(x)\,dx - (\theta'(t) + H_s(t,t))P(t) \\ P(t) &= -\int_t^{\pi/2} P'(x)\,dx \\ P'(t) &= -\int_t^{\pi/2} H_s(x,t)Q(x)\,dx - \int_t^{\pi/2} \left(\theta'(t)H_s(x,t) + \dfrac{\partial H_c(x,t)}{\partial t}\right)P(x)\,dx + H_c(t,t)P(t) \end{cases}. \tag{38}$$

If we denote that

$$F(t) = (Q(t), P(t), P'(t))^T$$

and

$$K(x,t)=\begin{pmatrix} H_c(x,t) & \theta'(t)H_c(x,t)-\dfrac{\partial H_s(x,t)}{\partial t} & -(\theta'(t)+H_s(t,t)) \\ 0 & 0 & 1 \\ H_s(x,t) & \theta'(t)H_s(x,t)+\dfrac{\partial H_c(x,t)}{\partial t} & H_c(t,t) \end{pmatrix},$$

equation (38) can be reduced to a vector form

$$F(t)+\int_t^{\pi/2}K(x,t)F(x)\,dx=0 \text{ for } 0<t<\frac{\pi}{2}. \tag{39}$$

Since the equation (39) is a homogenous Volterra integral equation, it only has the trivial solution. Therefore, we obtain

$$F(t)=\mathbf{0} \text{ for } 0<t<\frac{\pi}{2}$$

that gives us

$$Q(t)=P(t)=0 \text{ for } 0<t<\frac{\pi}{2}.$$

Thus, we obtain

$$p(x)=\widetilde{p}(x) \text{ and } q(x)=\widetilde{q}(x) \text{ on } (0,\pi).$$

Moreover, it is obvious that $h=\widetilde{h}$ from (36). □

Appendix

Substituting the functions $\varphi(x,\lambda)$ and $\varphi''(x,\lambda)$ instead of y and y'' in equation (1), respectively, we directly get following equalities

$$\omega(x)=xp(0)+\frac{2\rho(x)}{\beta^+}\int_0^x\left[A(\xi,\mu^+(\xi))\sin\tfrac{\omega(\xi)}{\sqrt{\rho(x)}}-B(\xi,\mu^+(\xi))\cos\tfrac{\omega(\xi)}{\sqrt{\rho(x)}}\right]d\xi,$$

$$p(x)=p(0)+\frac{2\alpha^2}{\beta^-}\left[A(x,t)\sin\tfrac{\omega(x)}{\sqrt{\rho(x)}}+B(x,t)\cos\tfrac{\omega(x)}{\sqrt{\rho(x)}}\right]\Bigg|_{t=\mu^-(x)-0}^{\mu^-(x)+0},$$

$$\beta^+\left[q(x)+\left(\tfrac{p(x)}{\sqrt{\rho(x)}}\right)^2\right]=2\sqrt{\rho(x)}\frac{d}{dx}\left[A(x,\mu^+(x))\cos\tfrac{\omega(x)}{\sqrt{\rho(x)}}+B(x,\mu^+(x))\sin\tfrac{\omega(x)}{\sqrt{\rho(x)}}\right],$$

$$\beta^-\left[q(x)+\left(\tfrac{p(x)}{\sqrt{\rho(x)}}\right)^2\right]=2\sqrt{\rho(x)}\frac{d}{dx}\left\{\left[A(x,t)\cos\tfrac{\omega(x)}{\sqrt{\rho(x)}}-B(x,t)\sin\tfrac{\omega(x)}{\sqrt{\rho(x)}}\right]\Bigg|_{t=\mu^-(x)-0}^{\mu^-(x)+0}\right\},$$

$$B(x,0)=\left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0}=0$$

and additionally if we suppose that $p(x) \in W_2^2(0,\pi)$, $q(x) \in W_2^1(0,\pi)$, then the functions $A(x,t)$ and $B(x,t)$ satisfy the following system of partial differential equations

$$\begin{cases} \dfrac{\partial^2 A(x,t)}{\partial x^2} - q(x)A(x,t) - 2p(x)\dfrac{\partial B(x,t)}{\partial t} = \rho(x)\dfrac{\partial^2 A(x,t)}{\partial t^2} \\ \dfrac{\partial^2 B(x,t)}{\partial x^2} - q(x)B(x,t) + 2p(x)\dfrac{\partial A(x,t)}{\partial t} = \rho(x)\dfrac{\partial^2 B(x,t)}{\partial t^2}. \end{cases}$$

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